Green's Function Theory of Multimode Cavities

G. KEMENY*
Ledgemont Laboratory, Kennecott Copper Corporation, Lexington, Massachusetts
(Received 1 April 1963; revised manuscript received 11 September 1963)

Schwinger has applied his generalized quantum action principle and the method of thermodynamic Green's functions to an harmonic oscillator. He has shown how this technique describes the buildup of cavity oscillations in a simple model of the maser. This paper applies Schwinger's technique to multimode cavities. The energy, autocorrelation (or coherence), and spectral distribution of individual cavity modes in the steady state, their rate of buildup and their response to external signals have been calculated. Conditions for the buildup and for the steady state of laser action are stated. It has been shown that the steady-state laser radiation in each cavity mode can be described in terms of spontaneous and induced emissions, the latter one containing a coherent and an incoherent part.

I. INTRODUCTION

THE technique of thermodynamic Green's functions has been applied extensively in the theories of the many-body problem and in quantum statistical mechanics.[1] A great advantage of this method is that it exhibits the structural relationship among various quantities very clearly. Thus, it provides an effective way to treat different aspects of a physical system by a unified approach. The dynamics of the system can be derived from Schwinger's action principle.[2,3]

Schwinger has applied these methods to a simple model of the maser.[3] He assumed a single lossless cavity mode to interact with a system of N two level atoms in complete resonance with each other. The spontaneous emission line shape of the atoms was given by a δ function. Independently of the maser problem, Schwinger has also treated the coupling of an oscillator to a loss mechanism.[3] One can relax the assumptions of the complete resonance and of the δ-function spontaneous emission line shape and also couple a set of oscillators to both the atoms and a loss mechanism, simultaneously. Then one has a model for multimode cavities. This is done in the present paper.

The application of the Green's function technique to this problem is logical. One is treating a many-body problem in which photons interact with atoms. Since one starts from first principles, intuitive extensions of concepts applicable to the radiation of single atoms are not necessary. The extension of such concepts ought to be justified. The separation of the radiation of the N-atom system into spontaneous and induced parts will be a natural consequence of the theory. The various quantities described in the abstract will be obtained as we go along without having to plan individual methods for their calculation.

The present work is related to that of Senitzky,[4] and of Wagner and Birnbaum.[5] These authors do not apply Schwinger's method, but share some of our interests. Senitzky's work is, however, restricted to single mode cavities. Wagner and Birnbaum only treat the steady-state radiation.

We thought it useful to briefly summarize Schwinger's technique in this paper. This is done in Secs. IV–VI.

II. THE MODEL OF THE MULTIMODE CAVITY

The transverse field in cavities can be represented in terms of harmonic oscillators. These harmonic oscillators are not free, but interact with two external systems. One of these systems consists of the resonant atoms which produce the cavity radiation. The other one is the loss mechanism. This may consist of the walls of the cavity and of the external space to which the radiation is coupled out.

The resonant atoms may have an arbitrary number of energy levels which are indirectly involved in the production of cavity radiation. We assume, however, that there are only two levels, an upper and a lower one, between which the radiative transitions take place.

The resonant atoms, besides being coupled to the oscillators, may also be coupled to one other system. This latter interaction may determine the electronic current autocorrelation in the atoms. We assume that this interaction provides an exponential relaxation mechanism for the current autocorrelation in the atoms, which leads to a Lorentzian spontaneous emission line shape. An example for such a mechanism is provided in the case of a solid-state laser by the interaction of the active impurity atoms with the host lattice. (If more than two levels are involved in the resonant radiative transitions, the Lorentzian spontaneous line shape can be masked by the energy distribution of upper and lower energy level groups.)

The spatial distribution of the resonant atoms is assumed to be random. If the coupling of the oscillators to the loss mechanism is not too strong, the oscillators can be treated as independent of each other.

* This work was done while the author was at American-Standard, Research Division, Union, New Jersey.

[1] See, e.g., Leo P. Kadanoff and Gordon Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1962), where references to the literature can be found.

[2] Julian Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452 (1951).

[3] Julian Schwinger, Brandeis University Summer Institute in Theoretical Physics, 1960 (unpublished); J. Math. Phys. **2**, 407 (1961).

[4] I. R. Senitzky, Phys. Rev. **127**, 1638 (1962). This paper refers to Senitzky's earlier work.

[5] W. G. Wagner and G. Birnbaum, J. Appl. Phys. **32**, 1185 (1961).

III. THE LAGRANGIAN OF THE FORCED ELECTROMAGNETIC OSCILLATORS

The transverse electromagnetic field in the cavity, represented by the vector potential ($\mathbf{A}$), interacts with the transverse components of the current ($\mathbf{j}_T$). This situation is described by the wave equation

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}(\mathbf{r},t)}{\partial t^2}-\nabla^2\mathbf{A}(\mathbf{r},t)=\frac{4\pi}{c}\mathbf{j}_T(\mathbf{r},t). \tag{3.1}$$

Both vector functions can be decomposed into orthonormal series:

$$\mathbf{A}(\mathbf{r},t)=c^2\sum_\lambda q_\lambda(t)f_\lambda(\mathbf{r})\mathbf{u}_\lambda \tag{3.2}$$

and

$$\mathbf{j}_T(\mathbf{r},t)=\frac{c}{4\pi}\sum_\lambda j_\lambda(t)f_\lambda(\mathbf{r})\mathbf{u}_\lambda, \tag{3.3}$$

where the summation over the wave vector λ is understood to include summation over the transverse polarizations. The $f_\lambda(\mathbf{r})$ are orthonormal functions which represent the cavity modes. $q_\lambda(t)$ and $j_\lambda(t)$ are the corresponding time amplitudes. The $\mathbf{u}_\lambda$ are unit vectors orthogonal to λ. The wave equation can be replaced now by

$$\ddot{q}_\lambda(t)+\omega_\lambda^2 q_\lambda(t)=j_\lambda(t). \tag{3.4}$$

There are two such equations for each wave vector.

The equations above do not take into account that the cavity is filled with a solid. This can be helped by multiplying the first term in Eq. (3.1) and the right-hand side in Eq. (3.3) by the square of the refractive index. In this way Eq. (3.4) becomes valid also for the solid.

The Lagrangian of the equations of motion for the forced electromagnetic oscillators is

$$L=\sum_\lambda[\tfrac{1}{2}(\dot{q}_\lambda^2-\omega_\lambda^2 q_\lambda^2)+q_\lambda j_\lambda(t)]. \tag{3.5}$$

There are various possibilities concerning $j_\lambda(t)$. It may be necessary to add the Lagrangian of the current carrying system to the one above and consider the entire dynamical system as one unit. The $j_\lambda(t)$ may also be known functions of time. Finally, the correlation functions of the $j_\lambda(t)$ may be known functions of time. We will only consider the last two possibilities.

Let us assume that there are two kinds of currents. One kind is a known function of time and its amplitudes will continue to be denoted by $j_\lambda(t)$. The other kind is determined by its known correlations and will be denoted by the $J_\lambda(t)$ time amplitudes. Thus the Lagrangian is

$$L=\sum_\lambda[\tfrac{1}{2}(\dot{q}_\lambda^2-\omega_\lambda^2 q_\lambda^2)+q_\lambda j_\lambda(t)+q_\lambda J_\lambda(t)]. \tag{3.6}$$

IV. THE QUANTUM ACTION PRINCIPLE

The general dynamical problem can be formulated and solved in the framework of Schwinger's quantum action principle which involves the technique of thermodynamic Green's-functions.[3] The transformation function for the closed time path $t_2\rightarrow t_1\rightarrow t_2$ is

$$\langle t_2|\,t_2\rangle_{\theta_0}=\prod_\lambda(1-e^{-\hbar\omega_\lambda\beta_0})\sum_{n_\lambda}\langle n_\lambda t_2|\,n_\lambda t_2\rangle^{F_\lambda\pm}e^{-n_\lambda\hbar\omega_\lambda\beta_0}, \tag{4.1}$$

where

$$\beta_0=1/kT_0=1/\theta_0. \tag{4.2}$$

This transformation function refers to the assembly of oscillators, each at the same temperature T_0 at the initial instant t_2. The external forces F_λ might be different in the forward transformation $t_2\rightarrow t_1$ from those in the backward transformation $t_1\rightarrow t_2$. These forces are $F_{\lambda+}$ and $F_{\lambda-}$, respectively. For the system under consideration these forces originate from the external currents. The n_λ represents the energy states of the oscillator by indicating the number of photons in it.

Let us supply the interaction terms in the Lagrangian with an adjustable factor α, such that for $\alpha=0$ we have free oscillators, and for $\alpha=1$ the actual physical situation is restored. Thus the Lagrangian appears in the form

$$L=\sum_\lambda[\tfrac{1}{2}(\dot{q}_\lambda^2-\omega_\lambda^2 q_\lambda^2)+\alpha q_\lambda j_\lambda(t)+\alpha q_\lambda J_\lambda(t)]. \tag{4.3}$$

The action principle then states that the variation of the transformation function over the closed time path is

$$\delta\langle t_2|\,t_2\rangle=\frac{i}{\hbar}\langle t_2|\,\delta\Big(\int_{t_2}^{t_1}dtL_+\Big)-\delta\Big(\int_{t_2}^{t_1}dtL_-\Big)|\,t_2\rangle, \tag{4.4}$$

where L_+ and L_- may differ because q_λ, j_λ, and J_λ can be different on the positive and negative segments of the transformation. If the variation is with respect to a then

$$\frac{\partial}{\partial a}\langle t_2|\,t_2\rangle_{\theta_0}{}^\alpha=\frac{i}{\hbar}\langle t_2|\int_{t_2}^{t_1}dt\prod_\lambda\{q_\lambda[j_\lambda(t)+J_\lambda(t)]|_+ -q_\lambda[j_\lambda(t)+J_\lambda(t)]|_-\}\,|\,t_2\rangle_{\theta_0}{}^\alpha. \tag{4.5}$$

The transformation function has to be evaluated for the initial thermal mixture and $\alpha=1$ will reproduce the actual physical system. One may try to approximate this expression by replacing the current time amplitudes by their expectation values. This is a satisfactory procedure for $j_\lambda(t)$ which is externally imposed and is a known function of time. $J_\lambda(t)$ is only statistically known and its expectation value is zero independent of t. (A possible additive constant would cause trivial complications.) The next approximation then leads to

$$\frac{\partial^2}{\partial a^2}\langle t_2|\,t_2\rangle_{\theta_0}{}^\alpha =-\frac{1}{\hbar^2}\langle t_2|\prod_\lambda\int_{t_2}^{t_1}dtdt'\{[q_\lambda J_\lambda(t)q_\lambda J_\lambda(t')]_{++} +[q_\lambda J_\lambda(t)q_\lambda J_\lambda(t')]_{--}-[q_\lambda J_\lambda(t)]_-[q_\lambda J_\lambda(t')]_+ -[q_\lambda J_\lambda(t')]_-[q_\lambda J_\lambda(t)]_+\}\,|\,t_2\rangle, \tag{4.6}$$

if the external currents $j_\lambda(t)$ are sufficiently weak to be neglected here. We will neglect crosscorrelations between modes and have, therefore, immediately replaced the double product by a single one. The consequences of this procedure will be examined in Secs. V and VI. The justification will follow in Sec. VII. The first term contains both t and t' on the forward segment of the transformation and is positively time ordered. The next term is negatively ordered because both times are on the negative segment. The last two terms are equal but symmetrically written. One time falls on the positive and the other one on the negative segment. The ordering is by the segment and not by the values of t and t' in these terms. The positive segment is first.

The expectation value of the second variation includes the current correlation components

$$A_{\lambda++}(t-t')=\frac{1}{\hbar}\langle[J_\lambda(t)J_\lambda(t')]_{++}\rangle, \tag{4.7a}$$

$$A_{\lambda--}(t-t')=\frac{1}{\hbar}\langle[J_\lambda(t)J_\lambda(t')]_{--}\rangle, \tag{4.7b}$$

$$A_{\lambda+-}(t-t')=\frac{1}{\hbar}\langle J_{\lambda-}(t')J_{\lambda+}(t)\rangle, \tag{4.7c}$$

$$A_{\lambda-+}(t-t')=\frac{1}{\hbar}\langle J_{\lambda-}(t)J_{\lambda+}(t')\rangle, \tag{4.7d}$$

which we assume to depend only on $(t-t')$.

The effective action operator, which reproduces the first and second derivatives of the transformation function with respect to α and the equations of motion of free oscillators for $\alpha=0$, is

$$\begin{aligned}W=\sum_\lambda\Big\{&\int_{t_2}^{t_1}dt[\tfrac{1}{2}(\dot q_\lambda{}^2-\omega_\lambda{}^2q_\lambda{}^2)+\alpha q_\lambda j_\lambda(t)|_+-\alpha q_\lambda j_\lambda(t)|_-]\\&+\frac{i}{2}\alpha^2\int_{t_2}^{t_1}\int_{t_2}^{t_1}dtdt'[q_\lambda(t)q_\lambda(t')]_+A_{\lambda++}(t-t')\\&+[q_\lambda(t)q_\lambda(t')]_-A_{\lambda--}(t-t')-q_{\lambda-}(t)q_{\lambda+}(t')\\&\times A_{\lambda-+}(t-t')-q_{\lambda-}(t')q_{\lambda+}(t)A_{\lambda+-}(t-t')\Big\}.\end{aligned} \tag{4.8}$$

Instead of utilizing the Lagrangian Eq. (3.6) the rest of this paper will be based upon the effective action operator. In replacing L by W a number of fundamental assumptions are made: (1) The external currents j_λ are so weak that it is sufficient to consider them only to first order in the coupling constant. (2) The $J_\lambda(t)$ currents are sufficiently weak to consider them only to second order. (3) The various current correlations A_λ are not appreciably effected by their coupling to the oscillators and depend only on $(t-t')$. The last assumption in particular is valid in solid-state systems and will be discussed in Sec. VII.

V. THE EQUATIONS OF MOTION. GREEN'S FUNCTIONS

The equations of motion which follow from the principle of stationary action are

$$\begin{aligned}\left(\frac{d^2}{dt^2}+\omega_0{}^2\right)q_+(t)-i\int_{t_2}^{t_1}dt'[&A_{++}(t-t')q_+(t')\\&-A_{+-}(t-t')q_-(t')]=j_+(t),\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}\left(\frac{d^2}{dt^2}+\omega_0{}^2\right)q_-(t)+i\int_{t_2}^{t_1}dt'[&A_{--}(t-t')q_-(t')\\&-A_{-+}(t-t')q_+(t')]=j_-(t).\end{aligned} \tag{5.2}$$

We have one such equation for each oscillator. The index denoting the oscillator has been dropped, the oscillator eigenfrequency is described by the generic notation ω_0 and $a=1$ has been substituted.

The thermal time boundary conditions at the initial time t_2 are

$$(q_++q_-)(t_2)=i/\omega_0\coth(\tfrac{1}{2}\beta_0\hbar\omega_0)\frac{d}{dt}(q_--q_+)|_{t=t_2}, \tag{5.3}$$

and

$$\frac{d}{dt}(q_++q_-)|_{t=t_2}=-i/\omega_0\coth(\tfrac{1}{2}\beta_0\hbar\omega_0)(q_--q_+)(t_2). \tag{5.4}$$

The difference variables do not necessarily vanish due to the difference between $j_+(t)$ and $j_-(t)$. The continuity of the two time segments at t_1 yields the boundary conditions

$$(q_--q_+)(t_1)=0, \tag{5.5}$$

and

$$\frac{d}{dt}(q_--q_+)|_{t=t_1}=0. \tag{5.6}$$

The equations of motion may also be expressed in terms of the difference and sum variables in the form

$$\begin{aligned}\left(\frac{d^2}{dt^2}+\omega_0{}^2\right)&(q_--q_+)(t)\\&-\int_{t_2}^{t_1}dt'A_a(t-t')(q_--q_+)(t')=(j_--j_+)(t),\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}\left(\frac{d^2}{dt^2}+\omega_0{}^2\right)&(q_-+q_+)(t)-\int_{t_2}^{t_1}dt'A_r(t-t')(q_++q_-)(t')\\&+i\int_{t_2}^{t_1}dt'a(t-t')(q_--q_+)(t')=(j_++j_-)(t),\end{aligned} \tag{5.8}$$

where

$$A_a(t-t')=i[A_{+-}(t-t')-A_{--}(t-t')], \quad (5.9)$$

$$A_r(t-t')=i[A_{++}(t-t')-A_{+-}(t-t')], \quad (5.10)$$

and

$$a(t-t')=A_{+-}(t-t')+A_{-+}(t-t') \quad (5.11)$$

are the advanced, retarded, and symmetrical current correlation functions, respectively. The solution of the difference equation can be exhibited in the form

$$(q_--q_+)(t)=\int_{t_2}^{t_1}dt'G_a(t-t')(j_--j_+)(t'), \quad (5.12)$$

where $G_a(t-t')$, the advanced Green's function, is the solution of the equation

$$\left(\frac{d^2}{dt^2}+\omega_0^2\right)G_a(t-t')-\int_{\infty}^{\infty}d\tau A_a(t-\tau)G_a(\tau-t')$$

$$=\delta(t-t') \quad \text{for} \quad t<t', \quad (5.13)$$

and

$$G_a(t-t')=0, \quad \text{for} \quad t>t'. \quad (5.14)$$

This satisfies the time boundary conditions at $t=t_1$. The retarded Green's function is defined by the equations

$$\left(\frac{d^2}{dt^2}+\omega_0^2\right)G_r(t-t')-\int_{-\infty}^{\infty}d\tau A_r(t-\tau)G_r(\tau-t')$$

$$=\delta(t-t') \quad \text{for} \quad t>t', \quad (5.15)$$

and

$$G_r(t-t')=0 \quad \text{for} \quad t<t'. \quad (5.16)$$

It can be shown that

$$G_r(t-t')=G_a(t'-t). \quad (5.17)$$

The solution of the sum equation which satisfies all the time boundary conditions of Eqs. (5.3)–(5.6) is

$$(q_++q_-)(t)$$

$$=\int_{t_2}^{t_1}dt'G_r(t-t')(j_++j_-)(t')$$

$$-i\int_{t_2}^{t_1}w(t-t_2,\,t'-t_2)(j_--j_+)(t')dt', \quad (5.18)$$

where

$$w(t-t_2,\,t'-t_2)$$

$$=\int_{t_2}^{\infty}\int_{t_2}^{\infty}d\tau d\tau' G_r(t-\tau)a(\tau-\tau')G_a(\tau'-t')$$

$$+\frac{1}{\omega_0}\coth(\tfrac{1}{2}\beta_0\hbar\omega_0)\left[\frac{\partial}{\partial t_2}G_r(t-t_2)\frac{\partial}{\partial t_2}G_a(t_2-t')\right.$$

$$\left.+\omega_0^2G_r(t-t_2)G_a(t_2-t')\right]. \quad (5.19)$$

Had the crosscorrelations between modes been included in Eq. (4.6), the A's in Eq. (4.7) would be matrices including nondiagonal elements. Equations (5.1), (5.2), (5.7), and (5.8) would show that the different modes are coupled. The Green's function Eqs. (5.13) and (5.15) would also show this property. Thus, the neglect of crosscorrelations reduces the multimode problem to a series of single-mode problems.

VI. EXPECTATION VALUES

If the expectation value of the quantity in (4.5) is written in terms of the sum and difference variables

$$\frac{\partial}{\partial\alpha}\langle t_2|\,t_2\rangle_{\theta_0{}^\alpha}=-\frac{i}{2\hbar}\langle t_2|\int_{t_2}^{t_1}dt\prod_\lambda[(j_{\lambda-}-j_{\lambda+})(q_{\lambda+}+q_{\lambda-})$$

$$+(j_{\lambda+}+j_{\lambda-})(q_{\lambda-}-q_{\lambda+})]|\,t_2\rangle \quad (6.1)$$

is obtained. Since crosscorrelations between modes are neglected, the transformation function of a set of oscillators is simply a product of the transformation functions of the individual oscillators. Thus, dropping the index λ, for each individual oscillator we have

$$\frac{\partial}{\partial\alpha}\langle t_2|\,t_2\rangle_{\theta_0{}^\alpha}=-\frac{i}{2\hbar}\langle t_2|\int_{t_2}^{t_1}dt[(j_--j_+)(t)(q_++q_-)(t)$$

$$+(j_++j_-)(t)(q_--q_+)(t)]|\,t_2\rangle. \quad (6.2)$$

Equations (5.12) and (5.18) have to be substituted here. Since the full coupling is not operative, both of these equations must be supplied with a factor α on the right-hand side. An integrable expression is obtained with Eq. (5.17) which leads to the integral

$$\langle t_2|\,t_2\rangle_{\theta_0}=\exp\left[-\frac{i}{2\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dtdt'(j_--j_+)(t)G_r(t-t')(j_++j_-)(t')\right]$$

$$\times\exp\left[-\frac{1}{4\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dtdt'(j_--j_+)(t)w(t-t')(j_--j_+)(t')\right] \quad (6.3)$$

for the effective transformation function, with $\alpha=1$. With the aid of Eqs. (5.12), (5.17) and (5.18) again

$$\langle t_2|\,t_2\rangle_{\theta_0}=\langle t_2|\int_{t_2}^{t_1}dte^{-(i/2\hbar)(j_--j_+)(t)(q_++q_-)(t)}$$

$$\times e^{-(i/2\hbar)(j_++j_-)(t)(q_--q_+)(t)}|\,t_2\rangle \quad (6.4)$$

is obtained. The first variational derivatives of the last two equations with respect to $(j_--j_+)(t)$ reproduce the two sides of Eq. (5.18). For $(j_--j_+)(t)=0$ the expectation value for the oscillator response to external current

$$q(t)=\int_{-\infty}^{\infty}G_r(t-t')j(t')dt' \quad (6.5)$$

is obtained. The second variational derivates of the same equations, evaluated also at $(j_- - j_+)(t)=0$ lead to the symmetrical autocorrelation[6] of the oscillator coordinate

$$[q(t),q(t')]_+=\hbar w(t-t_2,t'-t_2). \quad (6.6)$$

It is to be noted that the coupling of different modes would lead to crosscorrelations between the various modes.

VII. CORRELATIONS IN THE EXTERNAL SYSTEMS

The general method of the last three sections becomes applicable to multimode cavities if the external systems are identified with the resonant atoms and the loss mechanism, respectively. The retarded and symmetrical current autocorrelations in these systems must be determined.

Let us start with the atoms. The correlation functions required are of the form

$$\langle J_\lambda(t)J_{\lambda'}(t)\rangle=\left(\frac{4\pi}{c}\right)^2\int_V\int_V f_\lambda(\mathbf{r})f_{\lambda'}(\mathbf{r}') \times\langle(\mathbf{n}_\lambda\mathbf{J}_T(\mathbf{r},t))(\mathbf{n}_{\lambda'}\mathbf{J}_+(\mathbf{r}',t'))\rangle d^3r d^3r'. \quad (7.1)$$

If the atomic system is isotropic, the wavelength is long compared to the size of the atoms, the current correlations are localized within the randomly distributed atoms and different modes are uncorrelated, then

$$\langle J_\lambda(t)J_\lambda(t')\rangle=\left(\frac{4\pi}{c}\right)^2\frac{N}{V}\langle J_T(\mathbf{r},t)J_T(\mathbf{r},t')\rangle, \quad \text{if} \quad \lambda=\lambda'$$
$$=0, \quad \text{if} \quad \lambda\neq\lambda'. \quad (7.2)$$

N/V is the number of atoms per unit volume and the expectation value refers to the current correlation belonging to any one of the polarizations within any one of the atoms. Different modes are, in fact, uncorrelated. This follows from the random distribution of the atoms and the orthogonality of the modes.[5]

No attention has been paid to time ordering yet. The four possible orders of time were exhibited in Eq. (4.7). In order to construct $A_r(t-t')$ and $a(t-t')$ we seem to need three of these. Since

$$A_{++}(t-t')=A_{-+}(t-t') \quad \text{for} \quad t>t', \quad (7.3)$$

we may write

$$A_r(t-t')=i(A_{-+}(t-t')-A_{+-}(t-t')) \quad \text{for} \quad t>t',$$
$$=0 \quad \text{for} \quad t<t', \quad (7.4)$$

and

$$a(t-t')=A_{+-}(t-t')+A_{-+}(t-t'). \quad (7.5)$$

Thus it follows that A_{+-} and A_{-+} suffice. We have to evaluate them now. The calculation is described in Appendix 1. The physical picture is the following.

[6] The word "symmetrical" will later be dropped when it cannot lead to confusion.

The active atoms interact with two systems: (1) the solid, and (2) the cavity oscillators. Each atom in the solid suffers a thermal phonon collision about every 10^{-12} sec. This time is orders of magnitude shorter than the time of an electromagnetic transition. The phonons leave the average populations of the electronic states invariant, they only randomize the phases of the electronic state vectors very rapidly. The cavity oscillators have just the opposite effect. They cause transitions between the electronic states but change the phases of the electronic state vectors slowly. The effect of the transitions is offset by some external pumping mechanism which keeps the electronic state populations constant over the ensemble of active atoms. The phase changes caused by the oscillators are entirely negligible compared to the randomization of these phases by the phonons. Thus the cavity oscillators have no net effect on the atomic ensemble, as it was anticipated in Sec. IV. The atomic current autocorrelations can be calculated from the interaction of the atoms and the solid. The results are

$$A_{N+-}(t-t')=(1/\hbar)(4\pi/c)^2N/V|\langle|j_T|\rangle|^2[|a_u|^2e^{-i\Omega_1(t-t')} +|a_l|^2e^{i\Omega_1(t-t')}]e^{-\frac{1}{2}\Gamma|t-t'|}, \quad (7.6)$$

and

$$A_{N-+}(t-t')=(1/\hbar)(4\pi/c)^2N/V|\langle|j_T|\rangle|^2[|a_u|^2e^{i\Omega_1(t-t')} +|a_l|^2e^{-i\Omega_1(t-t')}]e^{-\frac{1}{2}\Gamma|t-t'|}. \quad (7.7)$$

The subscript N signifies that these quantities refer to the N atoms. The squared matrix element refers to the current transition matrix element for one polarization in the atom between the upper and lower levels, with amplitudes a_u and a_l, respectively. $|a_u|^2+|a_l|^2=1$ for two level atoms and possibly smaller for more levels. $|a_u|^2$ and $|a_l|^2$ are assumed to be constants. The energy difference of the two states is $\hbar\Omega_1$. The decay of the current correlations is exponential and has the reciprocal time constant $\Gamma/2$. The retarded and symmetrical correlation functions are

$$A_{rN}(t-t')=(i/\hbar)(4\pi/c)^2(N/V)|\langle|j|\rangle|^2(|a_u|^2-|a_l|^2) \times[e^{i\Omega_1(t-t')}+e^{-i\Omega_1(t-t')}] \times e^{-\frac{1}{2}\Gamma(t-t')} \quad \text{for} \quad t>t',$$
$$=0 \quad \text{for} \quad t<t', \quad (7.8)$$

and

$$a_N(t-t')=(1/\hbar)(4\pi/c)^2(N/V)|\langle|j|\rangle|^2(|a_u|^2+|a_l|^2) \times[e^{i\Omega_1(t-t')}+e^{-i\Omega_1(t-t')}]e^{-\frac{1}{2}\Gamma|t-t'|} \quad (7.9)$$

for the atomic current.

Under appropriate circumstances, specified by Schwinger,[3] the effect of the loss mechanism can be represented very simply. The retarded current correlation function is effectively equal to the damping constant γ multiplied by the time differential operator and is local in time.

$$A_{rL}(t-t')=\gamma\delta(t-t')d/dt' \quad \text{for} \quad t>t',$$
$$=0 \quad \text{for} \quad t<t', \quad (7.10)$$

where the subscript L indicates the loss mechanism. The symmetrical correlation function is

$$a_L(t-t')=a_L\delta(t-t'), \tag{7.11}$$

where

$$\gamma=\tfrac{1}{2}a_L[\tanh(\tfrac{1}{2}\hbar\omega_0\beta_L)/\omega_0] \tag{7.12}$$

and β_L is the temperature of the loss mechanism. These approximations describe the motion of the oscillators near their eigenfrequencies adequately. We note that γ and a_L are also generic notations and may have different values for different oscillators just as ω_0 has.

VIII. DETERMINATION OF THE RETARDED GREEN'S FUNCTION

The retarded current correlation $A_r(t-\tau)$, which appears in the equation for the retarded Green's function (5.15) is the sum of the corresponding quantities for the N atoms (7.8) and the loss mechanism (7.10).

$$\begin{aligned}A_r(t-\tau)&=(i/\hbar)(4\pi/c)^2N/V|\langle|j|\rangle|^2(|a_u|^2-|a_l|^2)\\&\times[e^{i\Omega_1(t-\tau)}+e^{-i\Omega_1(t-\tau)}]\\&\times e^{-\frac{1}{2}\Gamma(t-\tau)}+\gamma\delta(t-\tau)\frac{d}{d\tau},\quad \text{for}\quad t>\tau;\\&=0\quad \text{for}\quad t<\tau.\end{aligned} \tag{8.1}$$

The equation for the retarded Green's function (5.15) then appears by the substitution of (8.1) as

$$\begin{aligned}&\left(\frac{d^2}{dt^2}+\gamma\frac{d}{dt}+\omega_0^2\right)G_r(t-t')-\frac{i}{\hbar}\left(\frac{4\pi}{c}\right)^2\\&\times\frac{N}{V}|\langle|j|\rangle|^2(|a_u|^2-|a_l|^2)\\&\times\int_{-\infty}^{\infty}d\tau\eta_+(t-\tau)[e^{i\Omega_1(t-\tau)}+e^{-i\Omega_1(t-\tau)}]\\&\times e^{-\frac{1}{2}\Gamma(t-\tau)}G_r(\tau-t')=\delta(t-t')\quad\text{for}\quad t>t',\end{aligned} \tag{8.2}$$

and

$$G_r(t-t')=0\quad\text{for}\quad t<t'. \tag{8.2a}$$

The definition of $\eta_+(t-\tau)$ is

$$\begin{aligned}\eta_+(t-\tau)&=1\quad\text{for}\quad t>\tau,\\&=0\quad\text{for}\quad t<\tau.\end{aligned} \tag{8.3}$$

The solution of the differential equation can be obtained by Fourier transformation which is defined by

$$f(\zeta)=\int_{-\infty}^{\infty}d(t-t')e^{i\zeta(t-t')}f(t-t') \tag{8.4}$$

and

$$f(t-t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\zeta e^{-i\zeta(t-t')}f(\zeta). \tag{8.5}$$

Exponentially increasing functions of $(t-t')$ are allowed by supplying ζ with an appropriate positive imaginary part. The transform of Eq. (8.2) is

$$\left[-\zeta^2-i\gamma\zeta+\omega_0^2+\frac{\Omega_1^2\mu^2}{\Omega_1^2+(\Gamma/2)^2-i\Gamma\zeta-\zeta^2}\right]G(\zeta)=1, \tag{8.6}$$

where

$$\mu^2=(1/\hbar)(4\pi/c)^2(2N/V)|\langle|j|\rangle|^2(|a_u|^2-|a_l|^2)\frac{1}{\Omega_1}. \tag{8.7}$$

The quantity μ characterizes both the strength of the coupling of the atoms to the oscillators and also the inversion. The coupling of the oscillator to the two systems modifies its frequency to a small degree. We imagine that ω_0 embodies this change. We maintain the assumption that the upper and lower level occupation probabilities $|a_u|^2$ and $|a_l|^2$ are constants. $G(\zeta)$ can be expressed in the form

$$G(\zeta)=\left[-\zeta^2-i\gamma\zeta+\omega_0^2+\frac{\Omega_1^2\mu^2}{\Omega_1^2+(\Gamma/2)^2-i\Gamma\zeta-\zeta^2}\right]^{-1}. \tag{8.8}$$

We find in Appendix 2 that the four poles of $G(\zeta)$ are approximately given by

$$\begin{aligned}\zeta=&\pm\Omega-(i/2)\{\tfrac{1}{2}(\gamma+\Gamma)\\&\pm[\mu^2+(\gamma-\Gamma)^2/4-(\omega_1-\omega_0)^2]^{1/2}\},\end{aligned} \tag{8.9}$$

where

$$\Omega=+[\tfrac{1}{2}(\omega_0^2+\omega_1^2)]^{1/2} \tag{8.10}$$

and

$$\omega_1=+[\Omega_1^2+(\Gamma/2)^2]^{1/2}. \tag{8.11}$$

The approximation is based on the assumption that μ, γ, Γ and $|\omega_1-\omega_0|$ are all much smaller than ω_0 and ω_1.

The Fourier transformed Green's function can be exhibited in the factorized form

$$G(\zeta)=(\omega_1^2-i\Gamma\zeta-\zeta^2)/\prod_{i=1}^{4}(\zeta-\zeta_i), \tag{8.12}$$

where the ζi are the four poles. The inverse transform is determined by

$$G_r(t-t')=\frac{1}{2\pi}\oint d\zeta e^{-i\zeta(t-t')}(\omega_1^2-i\Gamma\zeta-\zeta^2)/\prod_{i=1}^{4}(\zeta-\zeta_i). \tag{8.13}$$

The integral should be nonzero for $t>t'$. The integrand will not diverge over the infinite semicircle completing the path if it is taken in the lower half of the complex ζ plane. Thus the contour integral runs counterclockwise parallel to the real axis above the poles and is closed by this semicircle. For $t<t'$ the advanced Green's function differs from zero. In this case the contour runs parallel to the real axis below the poles and the infinite semicircle lies in the upper half of the complex ζ plane. Using

Cauchy's theorem we obtain, approximately,

$$G_r(t-t')=\eta_+(t-t')\frac{\sin\Omega(t-t')}{2\Omega}\times\sum_{\pm\Gamma_\mu}\frac{\pm\Gamma_\mu+\Gamma_-}{\pm\Gamma_\mu}e^{(-\Gamma+\pm\Gamma_\mu)(t-t')}, \quad (8.14)$$

where

$$\Gamma_+=\tfrac{1}{4}(\Gamma+\gamma), \quad \Gamma_-=\tfrac{1}{4}(\Gamma-\gamma) \quad (8.15)$$

and

$$\Gamma_\mu=+[\Gamma_+^2+(\mu/2)^2-\tfrac{1}{4}\gamma\Gamma-(\omega_1-\omega_0)^2/4]^{1/2}. \quad (8.16)$$

The approximation in Eq. (8.14) is based on the same assumptions as those that led to Eq. (8.9). For $\omega_1=\omega_0$ and $\gamma=\Gamma=0$ the retarded Green's function becomes identical with the one given by Schwinger.[3] The retarded Green's function is related to a kernel used by Senitzky[7] with $\omega_1=\omega_0$ and $\Gamma=0$.

IX. THE RESPONSE OF AN ELECTROMAGNETIC OSCILLATOR TO CURRENT

The response of an electromagnetic oscillator to current was given in Eq. (6.5) in general terms. Suppose the current time amplitudes are Fourier analyzed and can be exhibited in the form

$$j(t)=\sum_{\Omega'}(j_{\Omega'}\sin\Omega' t+j_{\Omega'}'\cos\Omega' t) \quad \text{for} \quad t>0$$
$$=0 \quad \text{for} \quad t<0. \quad (9.1)$$

We compute the average response to one particular component, $\sin(\Omega' t)$, with the retarded Green's function given in Eq. (8.14). The average response is found to be

$$q(t)=\frac{j_{\Omega'}}{4\Omega}\sum_{\pm\Gamma_\mu}\frac{1}{(\Omega-\Omega')^2+(-\Gamma_+\pm\Gamma_\mu)^2}\frac{\pm\Gamma_\mu+\Gamma_-}{\pm\Gamma_\mu}\times\{(\Omega-\Omega')[\sin\Omega' t-\sin\Omega t e^{(-\Gamma+\pm\Gamma_\mu)t}]+(-\Gamma_+\pm\Gamma_\mu)[\cos\Omega' t-\cos\Omega t e^{(-\Gamma+\pm\Gamma_\mu)t}]\} \quad \text{for} \quad t>0$$
$$=0 \quad \text{for} \quad t<0. \quad (9.2)$$

This expression contains a Lorentzian resonance denominator as a function of the forcing frequency Ω'. As a function of the frequency Ω of the combined system, the denominator is not Lorentzian. Γ_μ and Γ_+ are functions of the parameters ω_0 and γ which vary from one oscillator to another. Thus we have a variable width.

The oscillator response contains two kinds of periodic terms with the frequencies Ω' and Ω, respectively. The latter kind contains two exponential factors. One of these can be built-up, constant or damped, depending on whether

$$\Gamma_\mu \gtrless \Gamma_+, \quad (9.3)$$

and the other one is always damped. This last relationship is equivalent to

$$\mu^2 \gtrless \gamma\Gamma+(\omega_1-\omega_0)^2. \quad (9.4)$$

A part of the response is thus built up if the strength of the atomic-oscillator coupling and the atomic inversion can overcome the joint effects of the atomic and oscillator dampings and of the imperfect resonance between the atoms and the oscillator. The built-up part of the response has the frequency of the combined atomic-oscillator system and not that of the forcing current. If there is no buildup, the response tends to a constant amplitude as $t\to\infty$ with the forcing frequency. Thus a steady state is reached. If the equality in the last relationship is satisfied one finds asymptotically that

$$q(t)=\frac{j_{\Omega'}}{2\Omega}\frac{\Gamma}{\Gamma+\gamma}\frac{1}{\Omega-\Omega'}(\sin\Omega' t-\sin\Omega t), \quad \text{for} \quad t\to\infty. \quad (9.5)$$

This contains only the $+\Gamma_\mu$ term. The $-\Gamma_\mu$ term would add a steady-state type of an expression. The part exhibited above oscillates with a small frequency at great amplitudes. For the resonant case $\Omega'=\Omega$ a linear increase

$$q(t)=(j_\Omega/2\Omega)(\Gamma/\Gamma+\gamma)t, \quad \text{for} \quad t\to\infty \quad (9.6)$$

is obtained.

The possibly very great response to external current at the frequency Ω is a consequence of the instability of the system at large inversions. This response is similar to the build-up obtained in the absence of external currents, as we will see in Sec. XI. There the buildup will be a consequence of current fluctuations in the atoms and in the loss mechanism and of the initial fluctuations in the cavity field. The rate of exponential buildup will be found there to be twice as great as it is here.

The response at the frequency Ω' of the current does not contain an exponential term but is purely periodic. Nevertheless, a great amount of energy can be coupled out of the cavity at this frequency under favorable circumstances. The amplified energy coupled out from a mode increases with γ and with the squares of the amplitudes of the periodic factors in Eq. (9.2). This γ includes the effect of both the wall losses and of the coupling out of the energy. γ can be increased arbitrarily and still get amplification as long as the amplitude factors do not decrease. This can be achieved by increasing the inversion to keep the balance. Note, however, that the amplitudes themselves cannot be arbitrarily increased by increasing the inversion. This would lead in the limit to $\Gamma_\mu\to\infty$ and the amplitude would vanish.

The buildup has to cease eventually when the populations get adjusted at such a level that $\Gamma_\mu<\Gamma_+$ for all modes. Then, after the transients have died out, only the Ω' frequency terms are present in the response and the amplification appears all by itself. If Γ_μ is only slightly smaller than Γ_+ only few oscillators, with

[7] I. R. Senitzky, Phys. Rev. **123**, 1525 (1961).

coupled system frequency Ω close to Ω', will respond to the external current.

X. CALCULATION OF THE OSCILLATOR AUTOCORRELATION FUNCTION

The oscillator autocorrelation can be computed by using Eqs. (6.6) and (5.19). G_r is given in Eq. (8.14). G_a is determined from Eq. (5.15). The symmetrical autocorrelation of the external systems is given in Eqs. (7.9) and (7.11). We will use the notation

$$a_N=(1/\hbar)(4\pi/c)^2N/V|\langle|j|\rangle|^2(|a_u|^2+|a_l|^2) \quad (10.1)$$

as the constant amplitude of the atomic correlation.

The oscillator autocorrelation can be exhibited as a sum of three terms which originate from the autocorrelations of the loss mechanism (w_L), of the atoms (w_N) and the initial oscillator autocorrelations (w_0), respectively.

$$w(t-t_2,\, t'-t_2)=w_L+w_N+w_0. \quad (10.2)$$

Cumbersome elementary calculations lead to the following lengthy results. The term originating from the loss mechanism is

$$w_L(t,t')=(a_L/16\Omega^2)\cos\Omega(t-t')\sum_{\pm\Gamma_\mu}\left\{\left(\frac{\pm\Gamma_\mu+\Gamma_-}{\pm\Gamma_\mu}\right)^2\frac{e^{(-\Gamma+\pm\Gamma_\mu)(t'-t)}-e^{(-\Gamma+\pm\Gamma_\mu)(t+t')}}{\Gamma_+\mp\Gamma_\mu}\right.$$
$$\left.+\frac{(\pm\Gamma_\mu)^2-\Gamma_-^2}{(\pm\Gamma_\mu)^2}\frac{e^{-\Gamma+(t'-t)}-e^{-\Gamma+(t+t')}}{\Gamma_+}e^{\pm\Gamma_\mu(t-t')}\right\};\quad \text{for}\quad t'\geq t\gg\Omega^{-1}. \quad (10.3)$$

The atomic contribution itself is best exhibited as a sum

$$w_N=w_{N++}+w_{N+-}+w_{N--}. \quad (10.4)$$

The subscripts refer to the signs of Γ_μ in this formula. w_{N++} contains both Γ_μ's with positive signs, w_{N--} has both with negative signs and w_{N+-} has them with mixed signs. We find

$$w_{N++}(t,t')+w_{N--}(t,t')=\frac{a_N}{16\Omega^2}\sum_{\pm\Gamma_\mu}\sum_{\pm\Omega}\left(\frac{\pm\Gamma_\mu+\Gamma_-}{\pm\Gamma_\mu}\right)^2e^{\pm i\Omega(t-t')}\left\{\frac{e^{(-\Gamma+\pm\Gamma_\mu)(t'-t)}-e^{(-\Gamma+\pm\Gamma_\mu)(t'+t)}}{2(\Gamma_+\mp\Gamma_\mu)}\right.$$
$$\times\left[\frac{1}{\mp i(\Omega_1-\Omega)+\Gamma_+\mp\Gamma_\mu+\Gamma/2}-\frac{1}{\mp i(\Omega-\Omega)+\Gamma_+\mp\Gamma_\mu-\Gamma/2}\right]$$
$$-\frac{e^{(-\Gamma+\pm\Gamma_\mu)t'}e^{[\pm i(\Omega_1-\Omega)-\Gamma/2]t}-e^{(-\Gamma+\pm\Gamma_\mu)(t+t')}}{[\pm i(\Omega_1-\Omega)+\Gamma_+\mp\Gamma_\mu-\Gamma/2][\mp i(\Omega_1-\Omega)+\Gamma_+\mp\Gamma_\mu+\Gamma/2]}$$
$$\left.+\frac{e^{[\mp i(\Omega_1-\Omega)-\Gamma/2](t'-t)}-e^{[\mp i(\Omega_1-\Omega)-\Gamma/2]t'}e^{(-\Gamma+\pm\Gamma_\mu)t}}{[\pm i(\Omega_1-\Omega)+\Gamma_+\mp\Gamma_\mu+\Gamma/2][\mp i(\Omega_1-\Omega)+\Gamma_+\mp\Gamma_\mu-\Gamma/2]}\right\};\quad t'\geq t\gg\Omega^{-1} \quad (10.5)$$

and

$$w_{N+-}(t,t')=\frac{a_N}{16\Omega^2}\frac{\Gamma_\mu^2-\Gamma_-^2}{\Gamma_\mu^2}\sum_{\pm\Gamma_\mu}\sum_{\pm\Omega}e^{\pm i\Omega(t-t')}\left\{\frac{e^{(\Gamma+\pm\Gamma_\mu)(t-t')}-e^{\pm\Gamma_\mu(t-t')}e^{-\Gamma+(t+t')}}{2\Gamma_+}\right.$$
$$\times\left[\frac{1}{\mp i(\Omega_1-\Omega)+\Gamma_+\pm\Gamma_\mu+\Gamma/2}-\frac{1}{\mp i(\Omega_1-\Omega)+\Gamma_+\pm\Gamma_\mu-\Gamma/2}\right]$$
$$+\frac{e^{[\mp i(\Omega_1-\Omega)-\Gamma/2](t'-t)}-e^{[\mp i(\Omega_1-\Omega)-\Gamma/2]t'}e^{(-\Gamma+\pm\Gamma_\mu)t}}{[\pm i(\Omega_1-\Omega)+\Gamma_+\mp\Gamma_\mu+\Gamma/2][\mp i(\Omega_1-\Omega)+\Gamma_+\mp\Gamma_\mu-\Gamma/2]}$$
$$\left.-\frac{e^{[+i(\Omega_1-\Omega)-\Gamma/2]t}e^{(-\Gamma+\mp\Gamma_\mu)t'}-e^{-\Gamma+(t+t')}e^{\mp\Gamma_\mu(t'-t)}}{[\pm i(\Omega_1-\Omega)+\Gamma_+\mp\Gamma_\mu-\Gamma/2][\mp i(\Omega_1-\Omega)+\Gamma_+\pm\Gamma_\mu+\Gamma/2]}\right\};\quad t'\geq t\gg\Omega^{-1}. \quad (10.6)$$

These expressions are real despite their complex form. We have chosen $t'\geq t$ even though really w_N is symmetrical in t and t'. For $t>t'$ the roles of t and t' must be interchanged. The summations over Ω and Γ_μ are independent of each other and allow four distinct sign combinations. Finally the original atomic autocorrela-

tion is given by

$$w_0(t,t')=\frac{\cos\Omega(t-t')}{4\Omega}\coth(\tfrac{1}{2}\beta_0\hbar\omega_0)\times\sum_{\pm\Gamma_\mu}\left\{\left(\frac{\pm\Gamma_\mu+\Gamma_-}{\pm\Gamma_\mu}\right)^2 e^{(-\Gamma_+\pm\Gamma_\mu)(t+t')}+\frac{\Gamma_\mu^2-\Gamma_-^2}{\Gamma_\mu^2}e^{-\Gamma_+(t+t')}e^{\pm\Gamma_\mu(t-t')}\right\};\quad t,t'\gg\Omega^{-1}. \tag{10.7}$$

We have neglected the difference between ω_0 and Ω in the last expression. We have everywhere taken the initial time $t_2=0$.

The oscillator autocorrelation must be investigated separately for the three possibilities shown by (9.3) or (9.4).

We will find that for $\Gamma_\mu\geq\Gamma_+$ the energy of an oscillator will increase in time (Sec. XI). This buildup cannot continue indefinitely. Eventually the atomic populations must adjust themselves such that $\Gamma_\mu<\Gamma_+$ and further buildup will cease. The atoms will still transfer energy to the oscillators but this will be compensated by the coupling to the loss mechanism. Thus a steady state with constant energy will be established (Sec. XII). When the autocorrelation for the steady state will be examined (Sec. XIII) it will be found that this quantity is damped. There is no contradiction here. The autocorrelation for the steady state depends on time only in the form $|t-t'|$. The energy is proportional to this quantity at $|t-t'|=0$ and thus it does not depend on time at all.

XI. BUILDUP OF THE OSCILLATOR ENERGY

The electromagnetic energy in one oscillator is given by

$$E=\Omega^2c^4\langle q^2(t)\rangle. \tag{11.1}$$

We find from Eq. (6.6) that

$$\langle q^2(t)\rangle=\tfrac{1}{2}\hbar w(t-t_2,t-t_2)\,|_{t_2=0}. \tag{11.2}$$

Thus $w(t,t)$ is proportional to the energy of the oscillator.

We will write down now the energy of the oscillator if $\Gamma_\mu>\Gamma_+$ long after the interactions with the external systems have been switched on. We are interested in the asymptotic limit $2(\Gamma_\mu-\Gamma_+)t\gg1$.

The loss mechanism term (10.3) in the oscillator autocorrelation yields

$$w_L(t,t)=\frac{a_L}{16\Omega^2}\left(\frac{\Gamma_\mu+\Gamma_-}{\Gamma_\mu}\right)^2\frac{e^{2(\Gamma_\mu-\Gamma_+)t}}{\Gamma_\mu-\Gamma_+};\quad 2(\Gamma_\mu-\Gamma_+)t\gg1. \tag{11.3}$$

Only w_{N++} out of the three parts of w_N leads to a term of comparable magnitude. Thus, asymptotically,

$$w_N(t,t)=\frac{a_N}{16\Omega^2}\left(\frac{\Gamma_\mu+\Gamma_-}{\Gamma_\mu}\right)^2 e^{2(\Gamma_\mu-\Gamma_+)t}\left[\frac{1}{\Gamma_\mu-\Gamma_+}\sum_{\pm}(\pm)\frac{\Gamma_+-\Gamma_\mu\pm\Gamma/2}{(\Omega_1-\Omega)^2+(\Gamma_+-\Gamma_\mu\pm\Gamma/2)^2}+2\prod_{\pm\Gamma}\frac{(\Omega_1-\Omega)^2+(\Gamma_+-\Gamma_\mu)^2-(\Gamma/2)^2}{(\Omega_1-\Omega)^2+(\Gamma_+-\Gamma_\mu\pm\Gamma/2)^2}\right];\quad 2(\Gamma_\mu-\Gamma_+)t\gg1. \tag{11.4}$$

Finally the initial thermal fluctuations lead to

$$w_0(t,t)=[\coth(\tfrac{1}{2}\hbar\beta_0\omega_0)/4\Omega](\Gamma_\mu+\Gamma_-)^2/\Gamma_\mu^2e^{2(\Gamma_\mu-\Gamma_+)t};\quad 2(\Gamma_\mu-\Gamma_+)t\gg1. \tag{11.5}$$

The sum of the last three equations is

$$w(t,t)=\frac{1}{16\Omega^2}\left(\frac{\Gamma_\mu+\Gamma_-}{\Gamma_\mu}\right)^2 e^{2(\Gamma_\mu-\Gamma_+)t}\left\{\frac{a_L}{\Gamma_\mu-\Gamma_+}+a_N\left[\frac{1}{\Gamma_\mu-\Gamma_+}\sum_{\pm}(\pm)\frac{\Gamma_+-\Gamma_\mu\pm\Gamma/2}{(\Omega_1-\Omega)^2+(\Gamma_+-\Gamma_\mu\pm\Gamma/2)^2}+2\prod_{\pm\Gamma}\frac{(\Omega_1-\Omega)^2+(\Gamma_+-\Gamma_\mu)^2-(\Gamma/2)^2}{(\Omega_1-\Omega)^2+(\Gamma_+-\Gamma_\mu\pm\Gamma/2)^2}\right]+4\Omega\coth(\tfrac{1}{2}\hbar\beta_0\omega_0)\right\};\quad 2(\Gamma_\mu-\Gamma_+)t\gg1. \tag{11.6}$$

We note, as before, that Γ_μ and Γ_+ are different for different oscillators. Therefore not all oscillator energies are built up and not all at the same rate. Besides the different rates of buildup, the constant amplitudes also bear the mark of resonance. The atomic term shows this property but it is naturally absent in the loss mechanism and original oscillator correlation terms. Since $\Gamma_\mu>\Gamma_+$ it follows that $\Gamma_\mu>\Gamma_-$ and the amplitude of the above expression does not vanish. It certainly is the leading asymptotic term. It is interesting that the three sources of the oscillator autocorrelation, namely the loss mechanism, the atomic and the original oscillator correlations, all contribute to the oscillator energy in the asymptotic limit. This is due to the strong buildup in the case $\Gamma_\mu>\Gamma_+$.

For the special case $\Gamma_\mu=\Gamma_+$ the energy increase is only

linear. We obtain

$$w(t,t)=\frac{t}{8\Omega^2}\left(\frac{\Gamma_\mu+\Gamma_-}{\Gamma_\mu}\right)^2\left[a_L+a_N\frac{\Gamma}{(\Omega_1-\Omega)^2+(\Gamma/2)^2}\right];$$
$$\Gamma_\mu=\Gamma_+,\quad \Gamma t\gg 1 \quad (11.7)$$

in the asymptotic limit. The initial oscillator correlation does not contribute this time.

Comparing the results of this section with Eqs. (9.3) and (9.4) the condition for buildup is

$$\Gamma_\mu\geq\Gamma_+ \quad (11.8)$$

or

$$\mu^2\geq\gamma\Gamma+(\omega_1-\omega_0)^2. \quad (11.9)$$

For the upper sign exponential, for the lower one linear buildup is obtained. Since μ^2 is proportional to the inversion according to Eq. (8.7), the eigenfrequency interval in which amplification occurs increases quadratically with the inversion. If $\mu^2<\gamma\Gamma$ there is no amplification at any frequency.

If the parameters ω_0 and γ for the various oscillators and ω_1 and Γ for the atoms are known and the rate of exponential buildup measured, μ^2 can be calculated. As time goes on μ^2 must decrease and only the less lossy and more resonant modes will continue to be built up, until finally the buildup stops entirely. μ^2 can be obtained as a function of time either by following which modes cease to take part in the buildup at a particular time or by measuring the energy of the least lossy and most resonant mode as a function of time. One may follow several modes simultaneously to check whether all of them can be described by the same μ^2. If the μ^2 prove to be different for different oscillators then the inversion has to be dependent on the modes. If this is the case the reaction of the oscillators on the atomic system has to be calculated.[8]

XII. OSCILLATOR ENERGY IN THE STEADY STATE

We consider the oscillator autocorrelation function for $\Gamma_\mu<\Gamma_+$. If we assume that $(\Gamma_+-\Gamma_\mu)(t+t')\gg 1$ and $|t-t'|\ll t+t'$ then, asymptotically, the oscillator autocorrelation becomes a function of only $|t-t'|$. For $t=t'$ this function is

$$w(t=t')=\frac{1}{16\Omega^2}\sum_{\pm\Gamma_\mu}\left\{\left[\left(\frac{\pm\Gamma_\mu+\Gamma_-}{\pm\Gamma_\mu}\right)^2\frac{1}{\Gamma_+\mp\Gamma_\mu}+\frac{(\pm\Gamma_\mu)^2-\Gamma_-^2}{(\pm\Gamma_\mu)^2}\frac{1}{\Gamma_+}\right]\right.$$
$$\times\left[a_L+a_N\left(\frac{\Gamma_+\mp\Gamma_\mu+\Gamma/2}{(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu+\Gamma/2)^2}-\frac{\Gamma_+\mp\Gamma_\mu-\Gamma/2}{(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu-\Gamma/2)^2}\right)\right]$$
$$+2a_N\left[\left(\frac{\pm\Gamma_\mu+\Gamma_-}{\pm\Gamma_\mu}\right)^2\frac{(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu)^2-(\Gamma/2)^2}{[(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu+\Gamma/2)^2][(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu-\Gamma/2)^2]}\right.$$
$$\left.\left.+\frac{(\pm\Gamma_\mu)^2-\Gamma_-^2}{(\pm\Gamma_\mu)^2}\frac{(\Omega_1-\Omega)^2+\Gamma_+^2+(\pm\Gamma_\mu-\Gamma/2)^2}{[(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu+\Gamma/2)^2][(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu-\Gamma/2)^2]}\right]\right\};\quad t'>t\gg(\Gamma_+-\Gamma_\mu)^{-1}. \quad (12.1)$$

Since $w(t=t')$ is independent of time a steady state is reached. The formula above determines the energy of the individual oscillators as a function of all the parameters involved. We will analyze this complicated expression. Let us assume that for the oscillators with the greatest value of Γ_μ the requirement

$$\Gamma_\mu<\Gamma_+ \quad (12.2)$$

is just barely satisfied, i.e., Γ_μ is very close to Γ_+. We would like to know then how $w(t=t')$ varies as a function of ω_0. We keep γ the same for all modes considered (for sake of argument). This will tell us about the energy content of the various oscillators in the cavity. It will be found that if Γ_μ is slightly smaller than Γ_+ for the oscillators with the greatest value of Γ_μ, steady-state laser action is achieved.

We have found that the inequality above can be written in the form

$$\mu^2<\gamma\Gamma+(\omega_1-\omega_0)^2. \quad (12.3)$$

The importance of this inequality is due to the presence of $(\Gamma_+-\Gamma_\mu)^{-1}$ in $w(t=t)$. This quantity can be exhibited as

$$(\Gamma_+-\Gamma_\mu)^{-1}=\{\Gamma_+-[\Gamma_+^2+(\mu/2)^2-\tfrac{1}{4}\gamma\Gamma-(\omega_1-\omega_0)^2/4]^{1/2}\}^{-1}. \quad (12.4)$$

The maximum as a function of the oscillator frequency is achieved at $\omega_0=\omega_1$, and here

$$(\Gamma_+-\Gamma_\mu)^{-1}=\{\Gamma_+-[\Gamma_+^2+(\mu/2)^2-\tfrac{1}{4}\gamma\Gamma]^{1/2}\}^{-1};\quad \text{if}\quad \omega_0=\omega_1. \quad (12.5)$$

Half of this value is obtained by solving the equation

$$(\Gamma_+-[\Gamma_+^2+(\mu/2)^2-\tfrac{1}{4}\gamma\Gamma-(\omega_1-\omega_0'/2)^2]^{1/2})^{-1}=\tfrac{1}{2}\{\Gamma_+-[\Gamma_+^2+(\mu/2)^2-\tfrac{1}{4}\gamma\Gamma]^{1/2}\}^{-1} \quad (12.6)$$

for ω_0'. Since $\Gamma_\mu\approx\Gamma_+$ we may expand both sides and obtain

$$(\omega_1-\omega_0)^2/4=\tfrac{1}{4}\gamma\Gamma-(\mu/2)^2. \quad (12.7)$$

[8] H. Haken and H. Sauermann, Z. Physik **173**, 261 (1963).

Thus the width of $(\Gamma_+-\Gamma_\mu)^{-1}$ as a function of the oscillator frequency ω_0 is

$$2|\omega_1-\omega_0'|=4[\gamma\Gamma-(\mu/2)^2]^{1/2}. \quad (12.8)$$

This shows that the energy content of the oscillators as a function of their eigenfrequency ω_0 can be made arbitrarily sharp by approaching the limit

$$(\mu/2)^2\rightarrow\gamma\Gamma \quad (12.9)$$

from the lower side. This function is centered at $\omega_0=\omega_1$. If

$$(\mu/2)^2=\gamma\Gamma, \quad (12.10)$$

a steady state is never reached for the perfectly resonant oscillator with $\omega_0=\omega_1$, because this last equation is equivalent to $\Gamma_\mu=\Gamma_+$. Then one of the requirements of the steady state, namely

$$(\Gamma_+-\Gamma_\mu)(t+t')\gg 1, \quad (12.11)$$

cannot be satisfied. The time necessary to establish the steady state diverges.

One can demonstrate that among all the functions which occur in $w(t=t')$ only $(\Gamma_+-\Gamma_\mu)^{-1}$ can be a sharp function of ω_0. Among all the quasi-Lorentzian half-widths only the following could vanish:

$$\Gamma_+-\Gamma_\mu-\Gamma/2=-\Gamma_\mu-\Gamma_- \quad (12.12)$$

and

$$\Gamma_++\Gamma_\mu-\Gamma/2=+\Gamma_\mu-\Gamma_-, \quad (12.13)$$

because $(\Gamma_+-\Gamma_\mu)$ and $(\Gamma_++\Gamma_\mu)$ are both positive numbers. When these widths vanish the amplitude of the terms in which they occur also vanish and thus these terms are entirely absent from Eq. (12.1).

The widths of the quasi-Lorentzians are not sensitive to the variations of Γ_μ in the neighborhood of Γ_+. These widths are about Γ for the $(\Gamma_+-\Gamma_\mu\pm\Gamma/2)$ and about $(4\Gamma_+\pm\Gamma/2)$ for the $(\Gamma_++\Gamma_\mu\pm\Gamma/2)$ terms. We see, then, that the width of the energy content function of the oscillators is identical with the width of $(\Gamma_+-\Gamma_\mu)^{-1}$.

We may, therefore, conclude that the only way to energize only the most resonant modes is by increasing the inversion until $(\mu/2)^2$ almost reaches the value $\gamma\Gamma$. Then the width of the energy content curve is $4[\gamma\Gamma-(\mu/2)^2]^{1/2}$. If

$$4[\gamma\Gamma-(\mu/2)^2]^{1/2}\ll\Gamma \quad (12.14)$$

is satisfied, where Γ is the width of spontaneous emission, laser action is achieved. We call this the energy content condition for steady-state laser action. A similar criterion was given by Wagner and Birnbaum.[5]

XIII. COHERENCE OF THE OSCILLATORS

According to the theory of partial coherence of Born and Wolf[9] the coherence of stationary fields is expressed by their autocorrelation function. The autocorrelation of a stationary field depends only on $|t-t'|$, by definition. This corresponds to our steady state. The definition of coherence has recently been the subject of much discussion. Coherence is related to correlation according to the definitions of Mandel and Wolf[10] and of Glauber.[11] Coherence is related to noise according to Senitzky.[12]

For the purposes of the present discussion it is not necessary to get involved in this controversy. In agreement with the definition of Born and Wolf, coherence in this paper refers to the ability of each mode to interfere with itself. This is expressed by the oscillator autocorrelation function.

The oscillator autocorrelation function for the steady state is the sum of the $w_L(t-t')$ and $w_N(t-t')$ terms. These expressions both contain exponentially decaying terms. The reciprocal time constants of decay are $(\Gamma_+-\Gamma_\mu)$, $\Gamma/2$ and $(\Gamma_++\Gamma_\mu)$. The terms are grouped accordingly in the following form of $w(t-t')$:

$$w(t-t')=\frac{1}{16\Omega^2}\sum_{\pm\Gamma_\mu}\Bigg(e^{(-\Gamma+\pm\Gamma_\mu)(t'-t)}\Bigg[\left(\frac{\pm\Gamma_\mu+\Gamma_-}{\pm\Gamma_\mu}\right)^2\frac{1}{\Gamma_+\mp\Gamma_\mu}+\frac{(\pm\Gamma_\mu)^2-\Gamma_-^2}{(\pm\Gamma_\mu)^2}\frac{1}{\Gamma_+}\Bigg]$$
$$\times\Bigg\{a_L\cos\Omega(t'-t)+a_N\Bigg[\frac{(\Omega_1-\Omega)\sin\Omega(t'-t)+(\Gamma_+\mp\Gamma_\mu+\Gamma/2)\cos\Omega(t'-t)}{(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu+\Gamma/2)^2}$$
$$-\frac{(\Omega_1-\Omega)\sin\Omega(t'-t)+(\Gamma_+\mp\Gamma_\mu-\Gamma/2)\cos\Omega(t'-t)}{(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu-\Gamma/2)^2}\Bigg]\Bigg\}+2e^{-\Gamma/2(t'-t)}a_N$$
$$\times\Bigg\{\left(\frac{\pm\Gamma_\mu+\Gamma_-}{\pm\Gamma_\mu}\right)^2\frac{[(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu)^2-(\Gamma/2)^2]\cos\Omega_1(t'-t)+\Gamma(\Omega_1-\Omega)\sin\Omega_1(t'-t)}{[(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu+\Gamma/2)^2][(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu-\Gamma/2)^2]}+\frac{(\pm\Gamma_\mu)^2-\Gamma_-^2}{(\pm\Gamma_\mu)^2}$$
$$\times\frac{[(\Omega_1-\Omega)^2+\Gamma_+^2+(\pm\Gamma_\mu-\Gamma/2)^2]\cos\Omega_1(t'-t)+(\Gamma_+\mp2\Gamma_\mu)(\Omega_1-\Omega)\sin\Omega_1(t'-t)}{[(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu+\Gamma/2)^2][(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu-\Gamma/2)^2]}\Bigg\}\Bigg);$$
$$t'\geq t\gg(\Gamma_++\Gamma_\mu)^{-1}. \quad (13.1)$$

[9] Max Born and Emil Wolf, *Optics* (Pergamon Press Inc., New York, 1959).
[10] L. Mandel and E. Wolf, J. Opt. Soc. Am. **51**, 815 (1961).
[11] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[12] I. R. Senitzky, Phys. Rev. **128**, 2864 (1962).

One of the three time constants, namely $(\Gamma/2)^{-1}$, is the same as the one for the decay of the atomic current autocorrelation function. This is the quantity which appears in spontaneous emission. Thus this term expresses the degree of coherence in the oscillator due to spontaneous emission. This part is considered to be incoherent. The amplitude of the spontaneous term depends on the inversion but its time constant does not.

The coherent radiation is described by the $(\Gamma_+-\Gamma_\mu)^{-1}$ time constant term. As $\Gamma_\mu \rightarrow \Gamma_+$ from smaller values the degree of coherence increases. For $\Gamma_\mu=\Gamma_+$ the coherence time becomes infinite but we have seen that it takes infinite length of time to establish such a steady state.

Finally the $(\Gamma_++\Gamma_\mu)^{-1}$ time constant term describes the possibly most incoherent part of the radiation. It may, in fact, be more incoherent than the spontaneous emission is. It is interesting to note that the more coherent the $(\Gamma_+-\Gamma_\mu)^{-1}$ part of the radiation becomes the more incoherent the $(\Gamma_++\Gamma_\mu)^{-1}$ part will be.

We note that the atomic autocorrelation contributes to all three time constant terms. The loss mechanism autocorrelation contributes only to the coherent and incoherent induced emissions but not to the spontaneous emission. This supports the interpretation of the oscillator autocorrelation function in terms of induced and spontaneous emissions. The spontaneous emission is due to a characteristic of the atoms, namely their current autocorrelation, which is assumed to be independent of the oscillators. Thus it is also independent of the oscillator-loss mechanism coupling. The induced emission depends on the oscillators themselves which are, in turn, influenced by both external systems. Additional support for this interpretation can be found in the fact that the spontaneous part of the oscillator autocorrelation decays at a rate independent of the oscillator frequency ω_0 and oscillator damping constant γ because Γ is an independent parameter. The $(-\Gamma_++\Gamma_\mu)^{-1}$ and $(-\Gamma_+-\Gamma_\mu)^{-1}$ factors in the exponents depend on the oscillator frequency through Γ_μ and on the oscillator damping constant γ through both Γ_+ and Γ_μ.

The last equation also shows that the most coherent term alone contains $(\Gamma_+-\Gamma_\mu)^{-1}$ in an amplitude factor. We have seen that $(\Gamma_+-\Gamma_\mu)^{-1}$ is a very sharp function of the oscillator frequency ω_0 if Γ_μ is almost as great as Γ_+ is. Thus the most coherent term occurs with the greatest amplitude. If this term is considered as a function of the oscillator frequency ω_0, it is found that as Γ_μ approaches Γ_+ from smaller values both the coherence and the amplitude of this term increase.

XIV. SPECTRAL DISTRIBUTION OF THE OSCILLATOR ENERGY

We have computed the energy content function of the oscillators $w(t=t')$ for the steady state before. In order to emphasize that this quantity depends, among other quantities, on the eigenfrequencies of the oscillators, we designate it now by $w(\omega_0)$. A particular oscillator does not, however, execute a purely harmonic motion at its eigenfrequency ω_0. This is due to the coupling of the oscillator to the two external systems. The energy of an oscillator can be analyzed into harmonic components of frequency ω. This analysis is expressed by the spectral distribution function $w(\omega)$. In the function $w(\omega)$ the ω_0 is a parameter. $w(\omega_0)$ then refers to the total energy of each oscillator as it varies from one oscillator to the next and $w(\omega)$ describes the amplitudes of the various frequency components ω of one particular oscillator with fixed ω_0.

The spectral distribution of the energy of the individual oscillators in the steady state can be found by Fourier transforming the corresponding autocorrelation functions. We wrote down this function for $t'>t$. It is symmetrical, because, in fact, it only depends on $|t-t'|$. Thus the spectral distribution is given by

$$w(\omega)=2\int_0^\infty d(t'-t)\cos\omega(t'-t)w(t'-t). \quad (14.1)$$

Substituting here Eq. (13.1) for $w(t'-t)$ we obtain

$$\begin{aligned}
w(\omega)=\frac{1}{16\Omega^2}\sum_{\pm\Gamma_\mu}\Bigg(&\frac{1}{(\Omega-\omega)^2+(\Gamma_+\mp\Gamma_\mu)^2}\left[\left(\frac{\pm\Gamma_\mu+\Gamma_-}{\pm\Gamma_\mu}\right)^2\frac{1}{\Gamma_+\mp\Gamma_\mu}+\frac{(\pm\Gamma_\mu)^2-\Gamma_-^2}{(\pm\Gamma_\mu)^2}\frac{1}{\Gamma_+}\right]\\
&\times\Bigg\{a_L(\Gamma_+\mp\Gamma_\mu)+a_N\Bigg[\frac{(\Omega_1-\Omega)(\Omega-\omega)+(\Gamma_+\mp\Gamma_\mu+\Gamma/2)(\Gamma_+\mp\Gamma_\mu)}{(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu+\Gamma/2)^2}\\
&-\frac{(\Omega_1-\Omega)(\Omega-\omega)+(\Gamma_+\mp\Gamma_\mu-\Gamma/2)(\Gamma_+\mp\Gamma_\mu)}{(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu-\Gamma/2)^2}\Bigg]\Bigg\}+\frac{2a_N}{(\Omega_1-\omega)^2+(\Gamma/2)^2}\\
&\times\Bigg\{\left[\frac{\pm\Gamma_\mu+\Gamma_-}{\pm\Gamma_\mu}\right]^2\frac{[(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu)^2-(\Gamma/2)^2]\Gamma/2+\Gamma(\Omega_1-\Omega)(\Omega_1-\omega)}{[(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu+\Gamma/2)^2][(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu-\Gamma/2)^2]}\\
&+\frac{(\pm\Gamma_\mu)^2-\Gamma_-^2}{(\pm\Gamma_\mu)^2}\frac{[(\Omega_1-\Omega)^2+\Gamma_+^2+(\pm\Gamma_\mu-\Gamma/2)^2]\Gamma/2+(\Gamma\mp2\Gamma_\mu)(\Omega_1-\Omega)(\Omega_1-\omega)}{[(\Omega_1-\Omega)^2+(\Gamma_+\mp\Gamma_\mu+\Gamma/2)^2][(\Omega_1-\Omega)^2+(\Gamma_+\pm\Gamma_\mu-\Gamma/2)^2]}\Bigg\}\Bigg).
\end{aligned} \quad (14.2)$$

The interpretation of this expression parallels that of the autocorrelation function. The first term in the large parenthesis belongs to the coherent and incoherent induced emissions. The second one describes the spontaneous emission. We note that the spectrum of the induced emissions is centered around the eigenfrequency of the coupled system Ω while that of the spontaneous emission is centered around the atomic frequency Ω_1.

Investigating the coherent induced emission term it is found that as $\Gamma_\mu \to \Gamma_+$ from the smaller side, by decreasing the difference $|\omega_0-\omega_1|$, not only the energy of the oscillator increases with $(\Gamma_+-\Gamma_\mu)^{-1}$ but also its spectral width becomes smaller [see Eq. (12.1)]. This means that an oscillator which is in closer resonance with the atoms is not only more highly energized but is also more monochromatic. This is exactly what makes laser action possible. In fact in the limit $\Gamma_\mu=\Gamma_+$ we extract

$$\lim_{\Gamma_+\to\Gamma_\mu} \frac{\Gamma_+-\Gamma_\mu}{(\Omega-\omega)^2+(\Gamma_+-\Gamma_\mu)^2}=\pi\delta(\Omega-\omega) \qquad (14.3)$$

from the first term, which means perfect monochromaticity. We mentioned before that in this limit the establishment of the steady state would require infinite length of time.

On the basis of the spectral distribution of a resonant oscillator we may say that it takes part in steady-state laser action if its spectrum is much narrower than it would be in spontaneous emission. Since the coherent part of the oscillation carries most of the energy, it essentially determines the line shape. Thus a condition for the participation of an oscillator in steady-state laser action may be formulated as

$$2(\Gamma_+-\Gamma_\mu)\ll\Gamma; \quad \Gamma_+>\Gamma_\mu. \qquad (14.4)$$

This oscillator then has a much narrower line shape in the steady-state laser operation than in the spontaneous emission. If the above inequality is written for Γ_μ just slightly smaller than Γ_+ it is equivalent to

$$2[\gamma\Gamma-(\mu/2)^2]^{1/2}\ll[\Gamma^2+\gamma\Gamma]^{1/2} \qquad (14.5)$$

for the case of $\omega_0=\omega_1$. We call this the spectral condition for steady-state laser action. This is essentially the same as the energy content condition where the energization of the oscillators with different eigenfrequencies is considered. In solid-state lasers $\gamma\ll\Gamma$ so that

$$2[\gamma\Gamma-(\mu/2)^2]^{1/2}\ll\Gamma \qquad (14.6)$$

and, apart of a factor of 2, this is the same condition as the one in (12.14).

The condition expressed by Eq. (14.4) or (14.5) could have been derived already in the previous section. There it would have been required that the coherence of the oscillator in steady-state laser action should be much greater than in spontaneous emission. The two formulations are equivalent because the spectral distribution is the Fourier transform of the autocorrelation function.

The width of the incoherent induced emission can take on the maximum value $(\Gamma+\gamma)$ when $\Gamma_+=\Gamma_\mu$. Thus the coherent induced emission has the narrowest spectral distribution and the incoherent one may be wider than the spontaneous emission which has the width Γ. This result is reminiscent to what one obtains in the Heitler damping theory.[13] External radiation with a narrow linewidth may stimulate resonant emission with a similarly narrow peak. This peak is superimposed on a background which is broader than that of the spontaneous emission because of the increase of the total transition probability.

The parallelism is imperfect because the Heitler damping theory refers to the natural linewidth which never appears in our calculation. Thus it happens only if the $(\Gamma_+-\Gamma_\mu)$ term is coherent enough that the $(\Gamma_++\Gamma_\mu)$ term becomes more incoherent than the spontaneous one.

The present spectral distribution can be compared with that of Wagner and Birnbaum.[5] The result of these authors cannot be interpreted as a superposition of coherent and incoherent induced emissions and of spontaneous emission because it contains only one term. They separate the atomic dipole moment into the sum of spontaneous and an induced moment, an assumption we do not have to make. Their cavity oscillators are driven by the spontaneous dipoles alone. In reality they are also driven by the loss mechanism.

It is interesting to analyze the spectral distribution function from a different point of view. One may ask about the variation of intensity of a particular fixed spectral component from oscillator to oscillator. Only those oscillators are considered for which Γ_μ is almost equal to Γ_+. Then

$$w(\omega)\sim\frac{1}{(\Omega-\omega)^2+(\Gamma_+-\Gamma_\mu)^2}\frac{1}{\Gamma_+-\Gamma_\mu} \qquad (14.7)$$

is an adequate representation of this dependence. As a function of ω this is a Lorentzian. For constant ω there are essentially two ranges of this function. If $|\Omega-\omega| \gg\Gamma_+-\Gamma_\mu$ then it varies like $1/(\Gamma_+-\Gamma_\mu)$. In this range then the energy in a frequency component varies from oscillator to oscillator proportionally to the variation of the total energy. If $|\Omega-\omega|\ll\Gamma_+-\Gamma_\mu$ then the energy in a frequency component varies like $1/(\Gamma_+-\Gamma_\mu)^3$. This is an extremely narrowly peaked function. It shows that the energy output of the laser at a particular frequency originates almost exclusively from oscillators which together with the atoms produce a combined system frequency Ω very close to ω. At ω_0', as determined from Eq. (12.7), the function $1/(\Gamma_+-\Gamma_\mu)^3$ has already fallen to $\frac{1}{8}$ of its maximum value. The energy content function at the same place has reached only $\frac{1}{2}$ its maximum value.

[13] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, Oxford, 1954), 3rd ed.

XV. MATHEMATICAL GENERALIZATIONS

A number of assumptions have been made which are fundamental to the mathematical approach of this paper. They are the following: (1) The currents are weak enough so that it is sufficient to go only to second order in the coupling constant. (2) The current correlations are practically unaffected by the coupling of the atoms to the oscillators. (3) The oscillators are independent of each other.

For the discussion of the buildup assumption (2) had to be slightly relaxed; namely, the atomic populations had to be allowed to be slow functions of time.

If the different oscillators are not independent of each other, crosscorrelations between different modes arise. This leads to crosscoherence between the modes. This possibility fits into the mathematical scheme of the paper but makes the calculations more difficult. The equations of motion of the modes do not separate but have to be solved simultaneously.

If the current correlations are significantly affected by the coupling of the atoms to the modes, which is the case in gas lasers, equations of motion for the atomic current have to be derived. The Lagrangian of Eq. (4.3) has to be completed by the addition of terms representing the atoms as dynamical systems.

If the currents are very strong, higher order terms in the coupling constant become important. This leads to the consideration of higher order correlations in the system. If the current correlations are unaffected by the coupling to the oscillators one may still derive an effective action operator, but this will be higher order than quadratic in the oscillator coordinates. The equations of motion will then be at least quadratic in the oscillator coordinates. The higher order correlations of the oscillator coordinates require a generalization of the concept of coherence, as discussed by Glauber.[11]

These generalizations all lead to considerable difficulties in carrying out actual calculations.

ACKNOWLEDGMENT

The author has benefited from many valuable discussions with Dr. P. J. Walsh and has enjoyed the encouragement of Dr. D. H. Howling.

APPENDIX

1. The Evaluation of the Atomic Current Autocorrelation

The atomic current autocorrelation in Eq. (7.2) is written in the Heisenberg picture. The current operators depend on time and the suppressed state vectors are constant. The evaluation of the expectation value is easier in the Dirac picture. In order to transform Eq. (7.2) into the Dirac picture the atomic Hamiltonian is split into two parts. One part denoted by V, describes the interaction between the atoms and the phonons. The other part contains the rest of the atomic Hamiltonian and includes the interaction with the cavity oscillators. If the two picture vectors coincide at the time t', then the relationship of the vectors and the operators in the two pictures are

$$|\psi_D(t)\rangle=\exp\left[-\frac{i}{\hbar}\int_{t'}^{t}V_D(\tau)d\tau\right]|\psi_H\rangle, \tag{A1}$$

$$F_D(t)=\exp\left[-\frac{i}{\hbar}\int_{t'}^{t}V_D(\tau)d\tau\right]F_H(t)\times\exp\left(\frac{i}{\hbar}\int_{t'}^{t}V_D(\tau)d\tau\right). \tag{A2}$$

With this choice of the transformation $|\psi_D(t)\rangle$ contains the time dependence due to the interaction between the atom and the phonons and $F_D(t)$ contains that between atom and cavity oscillators. The autocorrelation for an atom, as it can be seen from Eq. (7.2), is

$$\langle J(t)J(t')\rangle=\langle\psi_D(t)|J_D(t)\times\exp\left(-\frac{i}{\hbar}\int_{t'}^{t}V_D d\tau\right)J_D(t')|\psi_D(t')\rangle. \tag{A3}$$

With the introduction of the intermediate states m this becomes

$$\langle J(t)J(t')\rangle=\sum_m\langle\psi_D(t)|J_D(t)|\psi_{Dm}(t)\rangle\times\langle\psi_{Dm}(t')|J_D(t')|\psi_D(t')\rangle. \tag{A4}$$

The summation is extended over the upper and lower electron states. As we have mentioned in Sec. VII, the effect of the phonon collisions is a randomization of the phases of the electronic state vectors. Taking t' as the reference time the upper and lower state eigenvectors can be exhibited in the form

$$|\psi_{Du}(t)\rangle=[e^{-\frac{1}{2}\Gamma|t-t'|}+(1-e^{-\frac{1}{2}\Gamma|t-t'|})^{1/2}e^{i\delta'}]|\psi_{Du}(t')\rangle \tag{A5}$$

and

$$|\psi_{Dl}(t)\rangle=[e^{-\frac{1}{2}\Gamma|t-t'|}+(1-e^{-\frac{1}{2}\Gamma|t-t'|})^{1/2}e^{i\delta''}]|\Gamma_{Dl}(t')\rangle. \tag{A6}$$

$4/\Gamma$ is the time constant of phase randomization. δ' and δ'' are the randomized phases. The state vector of an atom is

$$|\psi_D(t)\rangle=a_l|\psi_{Dl}(t)\rangle+a_u|\psi_{Du}(t)\rangle, \tag{A7}$$

where a_l and a_u are constant complex amplitudes. The substitution of Eqs. (A5)–(A7) into (A4) leads to

$$\langle J(t)J(t')\rangle=|a_l|^2e^{-\frac{1}{2}\Gamma|t-t'|}\langle\psi_l|J(t)|\psi_u\rangle\langle\psi_u|J(t')|\psi_l\rangle+|a_u|^2e^{-\frac{1}{2}\Gamma|t-t'|}\langle\psi_u|J(t)|\psi_l\rangle\times\langle\psi_l|J(t')|\psi_u\rangle. \tag{A8}$$

The eigenvectors are always taken at the reference time and are therefore constants. In this formula the time ordering is such that $t>t'$ and thus $A_{++}(t-t')$ is being

calculated. If the energy difference between the upper and lower levels is taken to be $\hbar\Omega_1$ then, in view of Eqs. (7.2) and (7.3), Eq. (7.7) is obtained. The opposite time ordering leads to Eq. (7.6).

2. The Poles of the Fourier Transformed Green's Function

The poles of the Fourier transformed Green's function,

$$G(\zeta)=\left[-\zeta^2-i\gamma\zeta+\omega_0^2+\frac{\Omega_1^2\mu^2}{\Omega_1^2+(\Gamma/2)^2-i\Gamma\zeta-\zeta^2}\right]^{-1}$$

are the zeros of the bracketed expression. Exact and convenient solutions can be obtained only in a few special cases.

In general, the

$$(\zeta^2+i\gamma\zeta-\omega_0^2)(\zeta^2+i\Gamma\zeta-\omega_1^2)+\Omega_1^2\mu^2=0$$

quartic equation has to be solved, where

$$\omega_1^2=\Omega_1^2+(\Gamma/2)^2.$$

The solution can be attempted in the form

$$\zeta^2=\tfrac{1}{2}(\omega_0^2+\omega_1^2)+a,$$

where a is much smaller than the first term due to the weak coupling. Then

$$\zeta=\pm\left[\left(\frac{\omega_0^2+\omega_1^2}{2}\right)^{1/2}+\frac{a}{[2(\omega_0^2+\omega_1^2)]^{1/2}}\right].$$

By substitution into the quartic equation the

$$[a(1+i\gamma/2\Omega)+i\gamma\Omega+\tfrac{1}{2}(\omega_1^2-\omega_0^2)]\cdot[a(1+i\Gamma/2\Omega)+i\Gamma\Omega+\tfrac{1}{2}(\omega_0^2-\omega_1^2)]+\Omega_1^2\mu^2=0$$

quadratic equation is obtained with

$$\Omega^2=\tfrac{1}{2}(\omega_0^2+\omega_1^2).$$

One finds that

$$a\approx-\tfrac{1}{2}i(\gamma+\Gamma)\Omega\pm i[\Omega_1^2\mu^2+(\gamma-\Gamma)^2/4\Omega^2-(\omega_1^2-\omega_0^2)^2/4]^{1/2},$$

and thus

$$\zeta=\pm\Omega-\frac{i}{2}\left\{\frac{\gamma+\Gamma}{2}\pm\left[\mu^2+\left(\frac{\gamma-\Gamma}{2}\right)^2-\left(\frac{\omega_1^2-\omega_0^2}{2\Omega}\right)^2\right]^{1/2}\right\}.$$

In the neighborhood of the resonance $\omega_1+\omega_0\approx2\Omega$, so that

$$\zeta=\pm\Omega-\frac{i}{2}\{\tfrac{1}{2}(\gamma+\Gamma)\pm[\mu^2+(\gamma-\Gamma)^2/4-(\omega_1-\omega_0)^2]^{1/2}\}$$

determines the four poles with sufficient accuracy.